Homework Assignment #3: Answer Key

1. a. With the producer being the final user and with there being an unlimited amount of the resource, we simply want to max WTP - TC = $8q - .2q^2 - 2q$.
$\partial U/\partial q = 8 - .4q - 2 = 0$ ===> $q = 15$ per period, $p = 8 - .4(15) = 2$, $CS = 15(8-2)/2 = 45$, $\pi = 0$
Total welfare = CS + π = 45 + 0 = 45

Alternatively, you could realize that the optimal consumption in an unlimited world is the some as consumption under perfect competition as simply set P = MC and solve for q = 15.

b. We want to set the discounted marginal utility equal in every period.
Utility or Welfare = WTP - TC = $8q - .2q^2 - 2q$. MU = $\partial U/\partial q = 6 - .4q$

Using the substitution method
$MU_1 = 6 - .4q_1 = MU_2 = (6 - .4q_2) / (1.05)$
$q_1 + q_2 = 10$ ===> $q_1 = 10 - q_2$
$6 - .4(10 - q_2) = (6 - .4q_2) / (1.05)$
$.4q_2 + 6 - 4 = 5.71 - .38q_2$
$.78q_2 = 3.71$ ===> $q_2 = 4.76$, $q_1 = 5.24$, $p_2 = 6.10$, $p_1 = 5.90$, $CS_2 = 4.76(8 - 6.10)/2 = 4.52$, $CS_1 = 5.24(8 - 5.90)/2 = 5.50$, $\pi_2 = 4.76(6.10 - 2) = 19.52$, $\pi_1 = 5.24(5.90 - 2) = 20.44$.
Total welfare = (5.50 + 20.44) + (4.52 + 19.52)/1.05 = 25.94 + 22.90 = 48.84

Using the LaGrangian method
$\max U = 8q_1 - .2q_1^2 - 2q_1 + [8q_2 - .2q_2^2 - 2q_2]/(1+r) - \lambda(10 - q_1 - q_2)$
$\partial U/\partial q_1 = 8 - .4q_1 - 2 - \lambda = 0$
$\partial U/\partial q_2 = (8 - .4q_2 - 2)/(1+r) - \lambda = 0$
$\partial U/\partial \lambda = 10 - q_1 - q_2 = 0$
$8 - .4q_1 - 2 = (8 - .4q_2 - 2)/(1+r)$
$q_1 + q_2 = 10$ ===> $q_1 = 10 - q_2$
$8 - .4(10 - q_2) - 2 = (8 - .4q_2 - 2)/(1+.05)$
$2 + .4q_2 = (6 - .4q_2) / (1.05) = 5.71 - .38q_2$
$.78q_2 = 3.71$ ===> $q_2 = 4.76$, $q_1 = 5.24$ (as above)

c. With a monopolist we want to max π = TR - TC = $pq - 2q = (8 - .4q)q - 2q = 8q - .4q^2 - 2q$
$\partial \pi/\partial q = 8 - .8q - 2 = 0$ ====> $q = 7.5$, $p = 8 - .4(7.5) = 5$, $CS = (8 - 5)(7.5)/2 = 11.25$, $\pi = 5(7.5) - 2(7.5) = 22.5$. Total welfare = CS + π = 11.25 + 22.5 = 33.75

d. Using the LaGrangian method
$\max \pi = p_1q_1 - 2q_1 + [p_2q_2 - 2q_2]/(1+r) - \lambda(10 - q_1 - q_2)$
$\max \pi = (8 - .4q_1)q_1 - 2q_1 + [(8 - .4q_2)q_2 - 2q_2]/(1+r) - \lambda(10 - q_1 - q_2)$
$\partial U/\partial q_1 = 8 - .8q_1 - 2 - \lambda = 0$
$\partial U/\partial q_2 = (8 - .8q_2 - 2)/(1+r) - \lambda = 0$
$\partial U/\partial \lambda = 10 - q_1 - q_2 = 0$
$8 - .8q_1 - 2 = (8 - .8q_2 - 2)/(1+r)$
$q_1 + q_2 = 10$ ===> $q_1 = 10 - q_2$

$8 - .8(10 - q_2) - 2 = (8 - .8q_2 - 2)/(1+.05)$
$-2 + .8q_2 = (6 - .8q_2) / (1.05) = 5.71 - .76q_2$
$1.56q_2 = 7.71 \Longrightarrow q_2 = 4.94$, $q_1 = 5.06$, $p_2 = 6.02$, $p_1 = 5.98$, $CS_2 = 4.94(8 - 6.02)/2 = 4.89$, $CS_1 = 5.06(8 - 5.98)/2 = 5.11$, $\pi_2 = 4.94(6.02 - 2) = 19.86$, $\pi_1 = 5.06(5.98 - 2) = 20.14$.
Total welfare = (5.11 + 20.14) + (4.89 + 19.86)/1.05 = 25.25 + 23.57 = 48.82

e. If there is unlimited access to the resource, in each period firms enter until there are no profits. So in period 1, quantity is produced until P = MC or until the resource runs out. This occurs at $q_1 = 10$. In period 2, there is no resource left, so $q_2 = 0$. p_2 = n.a. $p_1 = 8 - .4(10) = 4$, $CS_1 = 10(8-4)/2 = 20$, $CS_2 = 0$, $\pi_1 = 10(4-2) = 20$, $\pi_2 = 0$. Total societal welfare = (20 + 20) + (0 + 0)/1.05 = 40 Note that the societal utility of monopoly (= 48.82) is close that of the social planner (= 48.84), but both far exceed that of perfect competition (= 40).

f. Rent seeking is unproductive behavior by firms that is costly and is designed to capture economic profits. Since perfect competitors in part e. make economic profits, these firms have the incentive to engage in costly behavior that would allow them to capture a larger share of the market. In the presence of rent-seeking, the social welfare of perfect competition would be even lower than that of the social planner because firm profits would be lower.

g. While we don't have an analytical method for determining the exact answer, we know that in the long-run we will move to a situation where annual consumption is 5 and the price = 6. The natural resource will be used to supplement this consumption until it is depleted.

h. See graphs from class.

2. a. The equation for the static reserve index is current proven reserves divided by current consumption. The static reserve index for U.S. petroleum is then 34/2.9 = 11.7 years.

b. There are many reasons why the static reserve index is overly alarmist about when the resource will finally be depleted.
1. Assumes no new deposits will be found.
2. Assumes no substitutes exist.
3. Assumes price stays constant as supply falls.
4. Assumes consumption stays constant in the face of falling supplies and rising prices.
5. Since price is constant also assumes that no new previously discovered resources finally become cost effective to extract.
6. Finally, the U.S. static reserve index only measures U.S. reserves. There may be plenty of imports to supply our needs even after domestic supplies are exhausted.

c. There is a very simple little equation for an annuity that grows by a set percentage over time.
$V = x[(1+r)^n - 1]/r$ where r is the growth rate, x is the year 1 consumption, n is the number of years of consumption, and V is the total amount of consumption over all the years. We know V = 34, x = 2.9, and r = .05, so we simply need to solve for n.

$34 = 2.9[(1.05)^n - 1]/.05$ ===> $n = 9.456$.

(Of course this overestimates the true time to depletion even more than the static reserve index.) This can also be approximated numerically using Excel. Cumulative consumption is 32.0 after 9 years and 36.5 after ten years, so the exponential reserve index must be between 9 and 10 years.

3. a. Using the LaGrangian method

$\max U = 8q_1 - .2q_1{}^2 - 2q_1 - 3q_1 + 3q_1 + [8q_2 - .2\, q_2{}^2 - 2q_2 - 3q_1 + 3q_1]/(1+r) - \lambda(10 - q_1 - q_2)$

$\partial U/\partial q_1 = 8 - .4q_1 - 2 - \lambda = 0$

$\partial U/\partial q_2 = (8 - .4q_2 - 2)/(1+r) - \lambda = 0$

$\partial U/\partial \lambda = 10 - q_1 - q_2 = 0$

$8 - .4q_1 - 2 = (8 - .4q_2 - 2)/(1+r)$

$q_1 + q_2 = 10$ ===> $q_1 = 10 - q_2$

$8 - .4(10 - q_2) - 2 = (8 - .4q_2 - 2)/(1+.05)$

$2 + .4q_2 = (6 - .4q_2\,) / (1.05) = 5.71 - .38q_2$

$.78q_2 = 3.71$ ===> $q_2 = 4.76$, $q_1 = 5.24$, $p_2 = 6.10$, $p_1 = 5.90$, $CS_2 = 4.76(8 - 6.10)/2 = 4.52$, $CS_1 = 5.24(8 - 5.90)/2 = 5.50$, $\pi_2 = 4.76(6.10 - 5) = 19.52$, $\pi_1 = 5.24(5.90 - 5) = 20.44$, $T_1 = 5.24(3)$, $T_2 = 4.76(3)$

Total welfare = (5.50 + 20.44) + (4.52 + 19.52)/1.05 = 25.94 + 22.90 = 48.84

Since tax revenues are counted as part of total welfare, the social planner's problem is unchanged from the situation in part 1. b.

b. Using the LaGrangian method

$\max \pi = p_1q_1 - 2q_1 - 3q_1 + [p_2q_2 - 2q_2 - 3q_1]/(1+r) - \lambda(10 - q_1 - q_2)$

$\max \pi = (8 - .4q_1)q_1 - 5q_1 + [(8 - .4q_2)q_2 - 5q_2]/(1+r) - \lambda(10 - q_1 - q_2)$

$\partial U/\partial q_1 = 8 - .8q_1 - 5 - \lambda = 0$

$\partial U/\partial q_2 = (8 - .8q_2 - 5)/(1+r) - \lambda = 0$

$\partial U/\partial \lambda = 10 - q_1 - q_2 = 0$ (if the constraint is binding)

$8 - .8q_1 - 5 = (8 - .8q_2 - 5)/(1+r)$

$q_1 + q_2 = 10$ ===> $q_1 = 10 - q_2$

$8 - .8(10 - q_2) - 5 = (8 - .8q_2 - 5)/(1+.05)$

$-5 + .8q_2 = (3 - .8q_2\,) / (1.05) = 2.86 - .76q_2$

$1.56q_2 = 7.86$ ===> $q_2 = 5.04$, $q_1 = 4.96$

At this point you should realize you have an error since we are consuming more in period 2 than in period 1. In fact, the problem is that the monopolist doesn't wish to produce all 10 units so the constraint is not binding.

With a single period monopolist we want to max $\pi = TR - TC = pq - 2q - 3q = (8 - .4q)q - 2q - 3q = 8q - .4q^2 - 2q - 3q$

$\partial \pi/\partial q = 8 - .8q - 2 - 3 = 0$ ====> $q = 3.75$, $p = 8 - .4(3.75) = 6.5$, $CS = (8 - 6.5)(3.75)/2 = 2.81$, $\pi = 6.5(3.75) - 2(3.75) - 3(3.75) = 5.63$, $T = 3(3.75) = 11.25$

Total welfare = CS + π + T = 2.81 + 5.63 + 11.25 = 19.69

The monopolist in a 2-period model with 10 units of the resource will produce 3.75 in each period. Total welfare = 19.69 + 19.69/1.05 = 19.69 + 18.75 = 38.44

c. If there is unlimited access to the resource, in each period firms enter until there are no profits. So in period 1, quantity is produced until P = MC + tax or until the resource runs out. This occurs at $p_1 = 5 = 8 - .4q_1 \implies q_1 = 7.5$. In period 2, we consume the rest, so $q_2 = 2.5$, $p_2 = 8 - .4(2.5) = 7$, $p_1 = 5$, $CS_1 = 7.5(8-5)/2 = 11.25$, $CS_2 = 2.5(8-7)/2 = 1.25$, $\pi_1 = 7.5(5-5) = 0$, $\pi_2 = 2.5(7-5) = 5$, $T_1 = 7.5(3) = 22.5$, $T_2 = 2.5(3) = 7.5$.
Total societal welfare = (11.25 + 0 + 22.5) + (1.25 + 5 + 7.5)/1.05 = 33.75 + 13.10 = 46.85

d. The presence of the tax doesn't affect societal welfare in the social planner's problem. It simply shifts benefits from firms to the government.

Society is made much worse off in the monopolist's problem. Without a severance tax, the monopolist's output closely matched the social planner's output. With a severance tax, the monopolist further reduces output to level significantly below the social planner in both periods.

The interesting example is in perfect competition. Total societal welfare is significantly increased with a severance tax. The tax artificially forces the perfect competitors to restrict output in period 1 leading to a result much closer to the social planner's desired result.

e. All one has to do is set the severance tax high enough such that the perfect competitors will charge a price equal to the first period price charged in the social planner's problem. A severance tax of 3.90 combined with the MC = 2 leads to $p_1 = 5.90 = 8 - .4q_1 \implies q_1 = 5.25$, the same as the social planner problem.